Section 13.6 – 14.1 Review Solutions

Section 13.6 Additional Exercises

1. The hyperboloid we are given has equation:

$$-x^2 - 9y^2 + 25z^2 = 1 \Leftrightarrow$$

$$x^2 + 9y^2 = 25z^2 - 1 \Leftrightarrow$$

$$\left(\frac{x}{1}\right)^2 + \left(\frac{y}{1/3}\right)^2 = \left(\frac{z}{1/5}\right)^2 - 1$$

From p. 688 we see that this is a two sheeted hyperboloid that does not contain any points whose z-coordinate satisfies $-1/5 < z < 1/5$.

So, the intersection is empty for $-1/5 < h < 1/5$.

For a fixed h outside this range, we have:

$$x^2 + \left(\frac{y}{1/3}\right)^2 = \underbrace{\left(\frac{h}{1/5}\right)^2 - 1}_{\text{constant}}$$

So, for h outside this range the intersection is an ellipse.

Section 13.7 Additional Exercises

1. Convert $(1, \pi/2, -2)$ from cylindrical to rectangular coordinates.

$$x = 1 \cdot \cos(\pi/2) = 0$$
$$y = 1 \cdot \sin(\pi/2) = 1$$
$$z = -2$$

$\boxed{\text{So, we get } (x, y, z) = (0, 1, -2)}$

2. Convert $(1, \sqrt{3}, 7)$ from rectangular to cylindrical coordinates.

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+\sqrt{3}^2} = \sqrt{4} = 2$$

$$\tan\theta = y/x = \sqrt{3}/1 = \sqrt{3} \Rightarrow$$

$$\theta = \pi/3 \quad \text{or} \quad 4\pi/3 .$$

The correct choice is $\theta = \pi/3$ because the projection $(1, \sqrt{3}, 0)$ lies in the first quadrant.

$\boxed{\text{So, we get } (r, \theta, z) = (2, \pi/3, 7)}$

3. Sketch the set $r = \sin\theta$ described in cylindrical coordinates.

If $r = 0$, then $\theta = 0$ or π and we get that $r = \sin\theta$ describes exactly the z-axis.

If $r \neq 0$, then using the relations

$r = \sqrt{x^2+y^2}$ and $\sin\theta = y/r$, we get

$$r = \sin\theta \iff$$

$$\sqrt{x^2+y^2} = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \iff$$

$$x^2+y^2 = y \iff$$

$$x^2+y^2-y = 0 \iff$$

$$x^2 + (y - 1/2)^2 = 1/4 \iff$$

$$x^2 + (y - 1/2)^2 = (1/2)^2$$

So, this describes a cylinder centered at $(0, -1/2)$ of radius $1/2$.

4. Convert $(3, 0, \pi/2)$ from spherical to rectangular coordinates.

$$x = \rho \sin\phi \cos\theta = 3 \cdot \sin(\pi/2) \cos(0) = 3$$
$$y = \rho \sin\phi \sin\theta = 3 \cdot \sin(\pi/2) \sin(0) = 0$$
$$z = \rho \cos\phi = 3 \cdot \cos(\pi/2) = 0$$

$\boxed{\text{So, we get } (x, y, z) = (3, 0, 0)}$

5. Convert $(1, 1, 1)$ from rectangular to spherical coordinates.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3}$$

$\tan\theta = \frac{y}{x} = 1 \rightarrow \theta = \pi/4$ or $5\pi/4$ since the projection $(1,1,0)$ is in 1^{st} Quadrant we must have $\theta = \pi/4$.

$$\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{3}}$$

Since $0 \le \phi \le \pi$ we must have $\phi = \cos^{-1}(1/\sqrt{3})$

$\approx .96$

$\boxed{\text{So, we get } (\rho, \theta, \phi) = (\sqrt{3}, \pi/4, .96)}$

6. Convert (2,0,2) from cylindrical to spherical coordinates.

Cylindrical $(r,\theta,z) \longrightarrow$ Spherical (ρ,θ,ϕ)

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\cos\phi = \frac{z}{\rho} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \phi = \pi/4$$

$$\theta = \theta = 0.$$

$\boxed{\text{So, we get } (2\sqrt{2},\ 0,\ \pi/4)}$

Section 14.1-14.2 Solutions

1. $r(t) = \langle \sin t, 0, 4+\cos t\rangle = \langle 0,0,4\rangle + \langle \sin t, 0, \cos t\rangle$

The circle has radius 1, it is centered at (0,0,4) and it is contained in the xz-plane.

2. Since the line is vertical its direction ~~of the~~ vector is parallel to the z-axis i.e. $d_v = \langle 0,0,1\rangle$. So, a parametrization is

$$\boxed{r(t) = \langle 3,2,0\rangle + t\langle 0,0,1\rangle = \langle 3,2,t\rangle,\ -\infty < t < \infty}$$

3. The circle of radius 2 with center (1,2,5) in a plane parallel to the yz-plane.

$$r(t) = \underbrace{\langle 1,2,5\rangle}_{\text{center}} + \langle 0, 2\cos t, 2\sin t\rangle$$

radius

we put $2\cos t$, $2\sin t$ in the y and z coordinates since the circle is parallel to the yz-plane.

$$\boxed{r(t) = \langle 1, 2\cos t + 2, 2\sin t + 5\rangle} \quad 0 \le t \le 2\pi$$

4. The intersection of the surfaces

$y^2 - z^2 = x - 2$ and $y^2 + z^2 = 9$, using $y = t$ as a parameter.

Setting $y = t \Rightarrow z = \pm\sqrt{9 - t^2}$

Plugging into the first equation gives:

$$t^2 - (9 - t^2) = x - 2 \Rightarrow$$

$2t^2 - 7 = x$, so we need two vector valued functions to parametrize the intersection ↳

$$\boxed{\begin{array}{l} r_1(t) = \langle 2t^2-7,\ t,\ \sqrt{9-t^2}\rangle \text{ and} \\ r_2(t) = \langle 2t^2-7,\ t,\ -\sqrt{9-t^2}\rangle \end{array}}$$

Note that we can parametrize the intersection by a single vector valued function using polar coordinates as on p. 708.

5. $r_1'(t) = \langle 2t, 3t^2, 1\rangle$.

6. $\frac{d}{dt}(r_1(t) \cdot r_2(t)) = r_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t)$

product rule for dot products

$$= \langle 2t, 3t^2, 1\rangle \cdot \langle e^{3t}, e^{2t}, e^t\rangle + \langle t^2, t^3, t\rangle \cdot \langle 3e^{3t}, 2e^{2t}, e^t\rangle$$

$$= 2te^{3t} + 3t^2e^{2t} + e^t + 3t^2e^{3t} + 2t^3e^{2t} + te^t$$

$$\boxed{= (2t+3t^2)e^{3t} + (3t^2+2t^3)e^{2t} + (1+t)e^t}$$

7. $\frac{d}{dt}(r_1(t) \times r_2(t)) = [r_1'(t) \times r_2(t)] + [r_1(t) \times r_2'(t)]$

[Product rule for cross products]

$$\boxed{= \langle 3t^2e^t - 2te^{2t} - e^{2t} + t^3e^t,\ e^{3t} + 3te^{3t} - t^2e^t - 2te^t,\ 2te^{2t} + 2t^2e^{2t} - 3t^2e^{3t} - 3t^3e^{3t} \rangle}$$

8. $\frac{d}{dt}(r \times r') = [r' \times r'] + [r \times r'']$

[product rule for cross product]

$= 0 + [r \times r'']$

[since $w \times v = 0$ iff $w = \lambda v$, $r' = r'$ so clearly $r' \times r' = 0$]

We have $r''(t) = \langle 0, 2, e^t \rangle$.

So, $r \times r'' = \begin{vmatrix} i & j & k \\ t & t^2 & e^t \\ 0 & 2 & e^t \end{vmatrix}$

$= (t^2e^t - 2e^t)i - (te^t)j + (2t)k$

$$= \boxed{\langle t^2e^t - 2e^t, -te^t, 2t \rangle}$$